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Controllability of Neutral Functional Integrodifferential Systems in Banach Spaces

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Abstract—Sufficient conditions for controllability of neutral functional integrodifferential systems in a Banach space are established. The results are obtained by using the Schaefer fixed-point theorem. An example is provided to illustrate the theory.

Keywords—Controllability, Neutral functional integrodifferential system, Schaefer fixed-point theorem.

1. INTRODUCTION

The problem of controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional space has been extensively studied. Several authors have extended the concept to infinite-dimensional systems in Banach spaces with bounded operators. Naito [1,2] has studied the controllability of semilinear systems, whereas Yamamoto and Park [3] discussed the same problem for parabolic equation with uniformly bounded nonlinear term. Controllability of nonlinear systems in abstract spaces has been studied by Chukwu and Lenhart [4]. Do [5] and Zhou [6] discussed the approximate controllability for a class of semilinear abstract equations. Kwun *et al.* [7] studied the approximate controllability for delay Volterra systems with bounded linear operators. Naito [8] established the controllability for nonlinear Volterra integrodifferential systems. Recently, Balachandran *et al.* [9,10] studied the controllability and local null controllability of Sobolev-type integrodifferential systems and functional differential systems in Banach spaces by using Schauder's fixed-point theorem. The purpose of this paper is to study the controllability of neutral functional integrodifferential systems in Banach spaces by using the Schaefer fixed-point theorem.

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2. PRELIMINARIES

Consider the neutral functional integrodifferential system of the form

$$\frac{d}{dt}[x(t) - g(t, x_t)] = Ax(t) + Bu(t) + \int_0^t f(s, x_s)\, ds, \qquad t \in J = [0, b], \qquad x_0 = \phi, \tag{1}$$

where the state $x(\cdot)$ takes values in the Banach space X and the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space. A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X, B is a bounded linear operator from U into X, $f : J \times C \to X$ and $g : J \times C \to X$ are continuous functions. Here $C = C([-r, 0], X)$ is the Banach space of all continuous functions $\phi : [-r, 0] \to X$ endowed with the norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}$. Also, for $x \in C([-r, b], X)$, we have $x_t \in C$ for $t \in [0, b]$, $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

We need the following fixed-point theorem due to Schaefer [11].

SCHAEFER THEOREM. *Let S be a convex subset of a normed linear space E and $0 \in S$. Let $F : S \to S$ be a completely continuous operator and let*

$$\zeta(F) = \{x \in S;\ x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

System (1) has a mild solution of the following form [12]:

$$\begin{aligned} x(t) &= T(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t AT(t - s) g(s, x_s)\, ds \\ &\quad + \int_0^t T(t - s) \left[(Bu)(s) + \int_0^s f(\tau, x_\tau)\, d\tau \right] ds, \qquad t \in J, \\ x_0 &= \phi. \end{aligned} \tag{2}$$

In order to study the controllability problem of (1), we introduce a parameter $\lambda \in (0, 1)$ and consider the following system [13]:

$$\frac{d}{dt}[x(t) - \lambda g(t, x_t)] = \lambda Ax(t) + \lambda Bu(t) + \lambda \int_0^t f(s, x_s)\, ds, \qquad t \in J = [0, b], \qquad x_0 = \phi. \tag{3}$$

Then the mild solution of (3) can be written as

$$\begin{aligned} x(t) &= \lambda T(t)[\phi(0) - g(0, \phi)] + \lambda g(t, x_t) + \lambda \int_0^t AT(t - s) g(s, x_s)\, ds \\ &\quad + \lambda \int_0^t T(t - s) \left[(Bu)(s) + \int_0^s f(\tau, x_\tau)\, d\tau \right] ds, \qquad t \in J, \\ x_0 &= \phi. \end{aligned}$$

DEFINITION. *System (1) is said to be controllable on the interval J if for every continuous initial function $\phi \in C$, there exists a control $u \in L^2(J, U)$ such that the solution $x(t)$ of (1) satisfies $x(b) = x_1$.*

We assume the following hypotheses.

(i) A is the infinitesimal generator of a compact semigroup of bounded linear operators T(t) in X such that

$$|T(t)| \le M_1, \quad \text{for some } M_1 \ge 1 \quad \text{and} \quad |AT(t)| \le M.$$

(ii) The linear operator $W : L^2(J, U) \to X$, defined by

$$Wu = \int_0^b T(b-s)Bu(s)\, ds,$$

has an invertible operator W^{-1}, which takes values in $L^2(J,U)/\ker W$ and there exist positive constants M_2, M_3 such that $|B| \le M_2$ and $|W^{-1}| \le M_3$.

(iii) For each $t \in J$, the function $f(t,.) : C \to X$ is continuous and for each $x \in C$, the function $f(.,x) : J \to X$ is strongly measurable.

(iv) For every positive integer k, there exists $\alpha_k \in L^1(0,b)$ such that

$$\sup_{|x| \le k} |f(t,x)| \le \alpha_k(t), \qquad \text{for } t \in J \quad \text{a.e.}$$

(v) The function g is completely continuous and such that the operator

$$G : C([-r,0],X) \to C([0,T],X)$$

defined by $(G\phi)(t) = g(t,\phi)$ is compact.

(vi) There exist constants $c_1 < 1$ and c_2 such that

$$|g(t,\phi)| \le c_1\|\phi\| + c_2, \quad t \in J, \quad \phi \in C.$$

(vii) There exists an integrable function $m : [0,b] \to [0,\infty)$ such that

$$|f(t,\phi)| \le m(t)\Omega(\|\phi\|), \qquad 0 \le t \le b, \quad \phi \in C,$$

where $\Omega : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(viii)

$$\int_0^b \hat{m}(s)\, ds \le \int_c^\infty \frac{ds}{s + \Omega(s)},$$

where

$$c = \frac{1}{1-c_1}\left[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + Mc_2 b + M_1 N b\right],$$

$$\hat{m}(t) = \max\left\{\frac{1}{1-c_1} M c_1, \frac{M_1}{M c_1} m(t)\right\}, \qquad \text{and}$$

$$N = M_2 M_3 \left[|x_1| + M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_1\|x_b\| + c_2 \right.$$
$$\left. + M\int_0^b (c_1\|x_s\| + c_2)\, ds + M_1 \int_0^t \int_0^s m(\tau)\Omega(\|x_\tau\|)\, d\tau\, ds\right].$$

3. MAIN RESULT

THEOREM. *If the hypotheses (i)–(viii) are satisfied, then system (1) is controllable on J.*

PROOF. Using hypothesis (ii) for an arbitrary function $x(\cdot)$, define the control

$$u(t) = W^{-1}\left[x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b, x_b) \right.$$
$$\left. - \int_0^b AT(b-s)g(s,x_s)\, ds - \int_0^b T(b-s)\int_0^s f(\tau, x_\tau)\, d\tau\, ds\right](t).$$

We shall now show that when using this control the operator defined by

$$Fx(t) = T(t)\,[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)\,ds + \int_0^t T(t-s)\left[(Bu)(s) + \int_0^s f(\tau,x_\tau)\,d\tau\right] ds, \qquad t \in J$$

has a fixed point. This fixed point is then a solution of equation (2).

Clearly, $(Fx)(b) = x_1$, which means that the control u steers the system from the initial function ϕ to x_1 in time b, provided we can obtain a fixed point of the nonlinear operator F.

First we obtain *a priori* bounds for the following equation:

$$\begin{aligned} x(t) &= \lambda T(t)[\phi(0) - g(0,\phi)] + \lambda g(t,x_t) + \lambda \int_0^t AT(t-s)g(s,x_s)\,ds \\ &+ \lambda \int_0^t T(t-\eta)BW^{-1}\left[x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b,x_b)\right. \\ &\left. - \int_0^b AT(b-s)g(s,x_s)\,ds - \int_0^b T(b-s)\int_0^s f(\tau,x_\tau)\,d\tau\,ds\right](\eta)\,d\eta \\ &+ \lambda \int_0^t T(t-s)\int_0^s f(\tau,x_\tau)\,d\tau\,ds. \end{aligned}$$

We have

$$\begin{aligned} |x(t)| &\le M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\|x_t\| + c_2 + M\int_0^t (c_1\|x_s\| + c_2)\,ds \\ &+ M_1 \int_0^t M_2M_3[|x_1| + M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\|x_b\| + c_2 \\ &+ M\int_0^b (c_1\|x_s\| + c_2)\,ds + M_1\int_0^b\int_0^s m(\tau)\Omega(\|x_\tau\|)\,d\tau\,ds]\,d\eta \\ &+ M_1\int_0^t\int_0^s m(\tau)\Omega(\|x_\tau\|)\,d\tau\,ds. \end{aligned}$$

We consider the function μ given by

$$\mu(t) = \sup\{|x(s)| : -r \le s \le t\}, \qquad 0 \le t \le b.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0,b]$ by the previous inequality, we have

$$\begin{aligned} \mu(t) &\le M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + Mc_1\int_0^{t^*} \mu(s)\,ds \\ &+ Mc_2b + M_1Nb + M_1\int_0^{t^*}\int_0^s m(\tau)\Omega(\mu(\tau))\,d\tau\,ds \\ &\le M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + Mc_1\int_0^t \mu(s)\,ds \\ &+ Mc_2b + M_1Nb + M_1\int_0^t\int_0^s m(\tau)\Omega(\mu(\tau))\,d\tau\,ds \end{aligned}$$

or

$$\begin{aligned} \mu(t) \le \frac{1}{1-c_1}\Bigg\{ & M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_2 + Mc_2b + M_1Nb \\ &+ Mc_1\int_0^t \mu(s)\,ds + M_1\int_0^t\int_0^s m(\tau)\Omega(\mu(\tau))\,d\tau\,ds\Bigg\}. \end{aligned}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M_1 \ge 1$.

Denoting by $v(t)$ the right-hand side of the above inequality, we have $c = v(0) = (1/(1-c_1)) \{M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_2 + +Mc_2 b + M_1 N b\}$, $\mu(t) \le v(t)$, $0 \le t \le b$, and

$$\begin{aligned} v'(t) &= \frac{1}{1-c_1} M c_1 \mu(t) + \frac{1}{1-c_1} M_1 \int_0^t m(\tau)\Omega(\mu(\tau))\, d\tau \\ &\le \frac{1}{1-c_1} M c_1 v(t) + \frac{1}{1-c_1} M_1 \int_0^t m(\tau)\Omega(v(\tau))\, d\tau \\ &\le \frac{1}{1-c_1} M c_1 \left\{ v(t) + \frac{M_1}{Mc_1} \int_0^t m(\tau)\Omega(v(\tau))\, d\tau \right\}. \end{aligned}$$

Let $w(t) = v(t) + (M_1/Mc_1)\int_0^t m(\tau)\Omega(v(\tau))\, d\tau$. Then $w(0) = v(0)$, $v(t) \le w(t)$, and

$$\begin{aligned} w'(t) &= v'(t) + \frac{M_1}{Mc_1} m(t)\Omega(v(t)) \\ &\le \frac{1}{1-c_1} M c_1 w(t) + \frac{M_1}{Mc_1} m(t)\Omega(w(t)) \\ &\le \hat{m}(t)\{w(t) + \Omega(w(t))\}. \end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{s+\Omega(s)} \le \int_0^b \hat{m}(s)\, ds < \int_c^\infty \frac{ds}{s+\Omega(s)}, \qquad 0 \le t \le b.$$

This inequality implies that there is a constant K such that $v(t) \le K$, $t \in [0,b]$, and hence, $\mu(t) \le K$, $t \in [0,b]$, $\|x_t\| \le \mu(t)$, we have

$$\|x\|_1 = \sup\{|x(t)| : -r \le t \le b\} \le K,$$

where K depends only on b and on the functions m and Ω.

In the second step, we rewrite problem (1) as follows. For $\phi \in C$ and define $\hat{\phi} \in C_b$, $C_b = C([-r,b],X)$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & -r \le t \le 0, \\ T(t)\phi(0), & 0 \le t \le b. \end{cases}$$

If $x(t) = y(t) + \hat{\phi}(t)$, $t \in [-r,b]$, it is easy to see that y satisfies

$$\begin{aligned} y_0 &= 0, \\ y(t) &= -T(t)g(0,\phi) + g\left(t, y_t + \hat{\phi}_t\right) + \int_0^t AT(t-s) g\left(s, y_s + \hat{\phi}_s\right) ds \\ &\quad + \int_0^t T(t-\eta) B W^{-1} \Bigg[x_1 - T(b)[\phi(0) - g(0,\phi)] - g\left(b, y_b + \hat{\phi}_b\right) \\ &\quad - \int_0^b AT(b-s) g\left(s, y_s + \hat{\phi}_s\right) ds - \int_0^b T(b-s) \int_0^s f\left(\tau, y_\tau + \hat{\phi}_\tau\right) d\tau\, ds \Bigg] (\eta)\, d\eta \\ &\quad + \int_0^t T(t-s) \int_0^s f\left(\tau, y_\tau + \hat{\phi}_\tau\right) d\tau\, ds \end{aligned}$$

if and only if x satisfies

$$\begin{aligned} x(t) &= T(t)[\phi(0) - g(0,\phi)] + g(t, x_t) + \int_0^t AT(t-s) g(s, x_s)\, ds \\ &\quad + \int_0^t T(t-\eta) B W^{-1} \Bigg[x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b, x_b) \\ &\quad - \int_0^b AT(b-s) g(s, x_s)\, ds - \int_0^b T(b-s) \int_0^s f(\tau, x_\tau)\, d\tau\, ds \Bigg] (\eta)\, d\eta \\ &\quad + \int_0^t T(t-s) \int_0^s f(\tau, x_\tau)\, d\tau\, ds \end{aligned}$$

and $x_0 = \phi$.

Define $C_b^0 = \{y \in C_b : y_0 = 0\}$ and $F : C_b^0 \to C_b^0$ by

$$(Fy)(t) = 0, \qquad -r \le t \le 0,$$

$$\begin{aligned}(Fy)(t) = &-T(t)g(0,\phi) + g\left(t, y_t + \hat{\phi}_t\right) + \int_0^t AT(t-s)\, g\left(s, y_t + \hat{\phi}_t\right) ds \\ &+ \int_0^t T(t-\eta)\, BW^{-1} \Bigg[x_1 - T(b)\left[\phi(0) - g(0,\phi)\right] - g\left(b, y_b + \hat{\phi}_b\right) \\ &- \int_0^b AT(b-s)\, g\left(s, y_s + \hat{\phi}_s\right) ds - \int_0^b T(b-s) \int_0^s f\left(\tau, y_\tau + \hat{\phi}_\tau\right) d\tau\, ds\Bigg](\eta)\, d\eta \\ &+ \int_0^t T(t-s) \int_0^s f\left(\tau, y_\tau + \hat{\phi}_\tau\right) d\tau\, ds, \qquad 0 \le t \le b.\end{aligned}$$

It will now be shown that F is a completely continuous operator.

Let $B_k = \{y \in C_b^0 : \|y\|_1 \le k\}$ for some $k \ge 1$. We first show that F maps B_k into an equicontinuous family. Let $y \in B_k$ and $t_1, t_2 \in [0,b]$. Then if $0 < t_1 < t_2 \le b$,

$$\begin{aligned}|(Fy)(t_1) - (Fy)(t_2)| \le\ &|T(t_1) - T(t_2)|\,|g(0,\phi)| + \left|g\left(t_1, y_{t_1} + \hat{\phi}_{t_1}\right) - g\left(t_2, y_{t_2} + \hat{\phi}_{t_2}\right)\right| \\ &+ \left|\int_0^{t_1} A[T(t_1-s) - T(t_2-s)]g\left(s, y_s + \hat{\phi}_s\right) ds\right| \\ &+ \left|\int_{t_1}^{t_2} AT(t_2-s)g\left(s, y_s + \hat{\phi}_s\right) ds\right| \\ &+ \Bigg|\int_0^{t_1} [T(t_1-\eta) - T(t_2-\eta)]\, BW^{-1} \Bigg[x_1 - T(b)\{\phi(0) - g(0,\phi\} \\ &- g\left(b, y_b + \hat{\phi}_b\right) - \int_0^b AT(b-s)g\left(s, y_s + \hat{\phi}_s\right) ds \\ &- \int_0^b T(b-s) \int_0^s f\left(\tau, y_\tau + \hat{\phi}_\tau\right) d\tau\, ds\Bigg](\eta)\, d\eta\Bigg| \\ &+ \Bigg|\int_{t_1}^{t_2} T(t_2-\eta)BW^{-1} \Bigg[x_1 - T(b)\{\phi(0) - g(0,\phi\} \\ &- g\left(b, y_b + \hat{\phi}_b\right) - \int_0^b AT(b-s)g\left(s, y_s + \hat{\phi}_s\right) ds \\ &- \int_0^b T(b-s) \int_0^s f\left(\tau, y_\tau + \hat{\phi}_\tau\right) d\tau\, ds\Bigg](\eta)\, d\eta\Bigg| \\ &+ \left|\int_0^{t_1} [T(t_1-s) - T(t_2-s)]f\left(s, y_s + \hat{\phi}_s\right) ds\right| \\ &+ \left|\int_{t_1}^{t_2} T(t_2-s)f\left(s, y_s + \hat{\phi}_s\right) ds\right| \\ \le\ &|T(t_1) - T(t_2)|\,|g(0,\phi)| + \left|g\left(t_1, y_{t_1} + \hat{\phi}_{t_1}\right) - g\left(t_2, y_{t_2} + \hat{\phi}_{t_2}\right)\right| \\ &+ \int_0^{t_1} |A[T(t_1-s) - T(t_2-s)]| \left(c_1 \left\|y_s + \hat{\phi}_s\right\| + c_2\right) ds \\ &+ \int_{t_1}^{t_2} |AT(t_2-s)| \left(c_1 \left\|y_s + \hat{\phi}_s\right\| + c_2\right) ds\Bigg| \\ &+ \int_0^{t_1} |T(t_1-\eta) - T(t_2-\eta)|\, M_2 M_3 \Bigg[\, |x_1| + M_1\{|\phi(0) - g(0,\phi)|\}\end{aligned}$$

$$
\begin{aligned}
&+c_1\left\|y_b+\hat{\phi}_b\right\|+c_2+M\int_0^b\left(c_1\left\|y_s+\hat{\phi}_s\right\|+c_2\right)ds\\
&\left.+M_1\int_0^b\int_0^s\alpha_{k'}(\tau)\,d\tau\,ds\right]d\eta\\
&+\int_{t_1}^{t_2}|T(t_2-\eta)|M_2M_3\left[|x_1|+M_1\{|\phi(0)-g(0,\phi)|\}\right.\\
&+c_1\left\|y_b+\hat{\phi}_b\right\|+c_2+M\int_0^b\left(c_1\left\|y_s+\hat{\phi}_s\right\|+c_2\right)ds\\
&\left.+M_1\int_0^b\int_0^s\alpha_{k'}(\tau)\,d\tau\,ds\right]d\eta\\
&+\int_0^{t_1}|T(t_1-s)-T(t_2-s)|\int_0^s\alpha_{k'}(\tau)\,d\tau\,ds\\
&+\int_{t_1}^{t_2}|T(t_2-s)|\int_0^s\alpha_{k'}(\tau)\,d\tau\,ds,
\end{aligned}
$$

where $k'=k+\|\hat{\phi}\|$. The right-hand side is independent of $y\in B_k$ and tends to zero as $t_2-t_1\to 0$, since g is completely continuous and the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology.

Thus, F maps B_k into an equicontinuous family of functions.

Notice that we considered here only the case $0<t_1<t_2$, since the other cases $t_1<t_2<0$ or $t_1<0<t_2$ are very simple.

It is easy to see that the family B_k is uniformly bounded. Next, we show $\overline{FB_k}$ is compact. Since we have shown FB_k is equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0<t\le b$ be fixed and ϵ a real number satisfying $0<\epsilon<t$. For $y\in B_k$, we define

$$
\begin{aligned}
(F_\epsilon y)(t)&=-T(t)g(0,\phi)+g\left(t-\epsilon,y_{t-\epsilon}+\hat{\phi}_{t-\epsilon}\right)+\int_0^{t-\epsilon}AT(t-s)g\left(s,y_s+\hat{\phi}_s\right)ds\\
&\quad+\int_0^{t-\epsilon}T(t-\eta)BW^{-1}\left[x_1-T(b)[\phi(0)-g(0,\phi)]-g\left(b,y_b+\hat{\phi}_b\right)\right.\\
&\quad\left.-\int_0^b AT(b-s)g\left(s,y_s+\hat{\phi}_s\right)ds-\int_0^b T(b-s)\int_0^s f\left(\tau,y_\tau+\hat{\phi}_\tau\right)d\tau\,ds\right](\eta)\,d\eta\\
&\quad+\int_0^{t-\epsilon}T(t-s)\int_0^s f\left(\tau,y_\tau+\hat{\phi}_\tau\right)d\tau\,ds\\
&=-T(t)g(0,\phi)+g(t-\epsilon,y_{t-\epsilon})+\hat{\phi}_{t-\epsilon}+T(\epsilon)\int_0^{t-\epsilon}AT(t-s-\epsilon)g\left(s,y_s+\hat{\phi}_s\right)ds\\
&\quad+T(\epsilon)\int_0^{t}T(t-\eta-\epsilon)BW^{-1}\left[x_1-T(b)[\phi(0)-g(0,\phi)]-g\left(b,y_b+\hat{\phi}_b\right)\right.\\
&\quad\left.-\int_0^b AT(b-s)g\left(s,y_s+\hat{\phi}_s\right)ds-\int_0^b T(b-s)\int_0^s f\left(\tau,y_\tau+\hat{\phi}_\tau\right)d\tau\,ds\right](\eta)\,d\eta\\
&\quad+T(\epsilon)\int_0^{t-\epsilon}T(t-s-\epsilon)\int_0^s f\left(\tau,y_\tau+\hat{\phi}_\tau\right)d\tau\,ds.
\end{aligned}
$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t)=\{(F_\epsilon y)(t):y\in B_k\}$ is precompact in X for every ϵ, $0<\epsilon<t$. Moreover, for every $y\in B_k$ we have

$$
\begin{aligned}
&|(Fy)(t) - (F_\epsilon y)(t)| \\
&\le \left|g\left(t, y_t + \hat{\phi}_t\right) - g\left(t-\epsilon, y_{t-\epsilon} + \hat{\phi}_{t-\epsilon}\right)\right| + \int_{t-\epsilon}^{t} \left|AT(t-s) g\left(s, y_s + \hat{\phi}_s\right)\right| ds \\
&\quad + \int_{t-\epsilon}^{t} \Bigg| T(t-\eta) BW^{-1} \Bigg[x_1 - T(b)\left[\phi(0) - g(0,\phi)\right] - g\left(b, y_b + \hat{\phi}_b\right) \\
&\quad - \int_0^b AT(b-s) g\left(s, y_s + \hat{\phi}_s\right) ds - \int_0^b T(b-s) \int_0^s f\left(\tau, y_\tau + \hat{\phi}_\tau\right) d\tau\, ds \Bigg](\eta) \Bigg| d\eta \\
&\quad + \int_{t-\epsilon}^{t} \left| T(t-s) \int_0^s f\left(\tau, y_\tau + \hat{\phi}_\tau\right) d\tau \right| ds \\
&\le \left|g\left(t, y_t + \hat{\phi}_t\right) - g\left(t-\epsilon, y_{t-\epsilon} + \hat{\phi}_{t-\epsilon}\right)\right| + \int_{t-\epsilon}^{t} \left|AT(t-s) g\left(s, y_s + \hat{\phi}_s\right)\right| ds \\
&\quad + \int_{t-\epsilon}^{t} |T(t-\eta)|\, M_2 M_3 \Bigg[|x_1| + M_1 |\phi(0) - g(0,\phi)| + \left|g\left(b, y_b + \hat{\phi}_b\right)\right| \\
&\quad + M \int_0^b \left|g\left(s, y_s + \hat{\phi}_s\right)\right| ds + M_1 \int_0^b \int_0^s \alpha_{k'}(\tau)\, d\tau\, ds \Bigg] d\eta \\
&\quad + \int_{t-\epsilon}^{t} |T(t-s)| \int_0^s \alpha_{k'}(\tau)\, d\tau\, ds.
\end{aligned}
$$

Therefore, there are precompact sets arbitrarily close to the set $\{(Fy)(t) : x \in B_k\}$. Hence, the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X.

It remains to show that $F : C_b^0 \to C_b^0$ is continuous. Let $\{y_n\}_0^\infty \subseteq C_b^0$ with $y_n \to y$ in C_b^0. Then there is an integer r such that $|y_n(t)| \le r$ for all n and $t \in J$, so $y_n \in B_r$ and $y \in B_r$. By (iii), $f(t, y_n(t) + \hat{\phi}_t) \to f(t, y(t) + \hat{\phi}_t)$ for each $t \in J$ and since $|f(t, y_n(t) + \hat{\phi}_t) - f(t, y(t) + \hat{\phi}_t)| \le 2\alpha_{r'}(t)$, $r' = r + \|\hat{\phi}\|$ and also g is completely continuous, we have by the dominated convergence theorem

$$
\begin{aligned}
\|Fy_n - Fy\| &= \sup_{t \in J} \Bigg| \left[g\left(t, y_n(t) + \hat{\phi}_t\right) - g\left(t, y(t) + \hat{\phi}_t\right)\right] + \int_0^t AT(t-s) \Big[g\left(s, y_n(s) + \hat{\phi}_s\right) \\
&\quad - g\left(s, y(s) + \hat{\phi}_s\right)\Big] ds \\
&\quad + \int_0^t T(t-\eta) BW^{-1} \Bigg[\int_0^b AT(b-s) \left[g\left(s, y_n(s) + \hat{\phi}_s\right) - g\left(s, y(s) + \hat{\phi}_s\right)\right] ds \\
&\quad + \int_0^b T(b-s) \int_0^s \left[f\left(\tau, y_n(\tau) + \hat{\phi}_\tau\right) - f\left(\tau, y(\tau) + \hat{\phi}_\tau\right)\right] d\tau\, ds \Bigg](\eta)\, d\eta \\
&\quad + \int_0^t T(t-s) \int_0^s \left[f\left(\tau, y_n(\tau) + \hat{\phi}_\tau\right) - f\left(\tau, y(\tau) + \hat{\phi}_\tau\right)\right] d\tau\, ds \Bigg| \\
&\le \left|g\left(t, y_n(t) + \hat{\phi}_t\right) - g\left(t, y(t) + \hat{\phi}_t\right)\right| + \int_0^b |AT(t-s)| \Big| g\left(s, y_n(s) + \hat{\phi}_s\right) \\
&\quad - g\left(s, y(s) + \hat{\phi}_s\right)\Big| ds \\
&\quad + \int_0^b |T(t-\eta)|\, M_2 M_3 \Bigg[M \int_0^b \left|g\left(s, y_n(s) + \hat{\phi}_s\right) - g\left(s, y(s) + \hat{\phi}_s\right)\right| ds \\
&\quad + M_1 \int_0^b \int_0^s \left|f\left(\tau, y_n(\tau) + \hat{\phi}_\tau\right) - f\left(\tau, y(\tau) + \hat{\phi}_\tau\right)\right| d\tau\, ds \Bigg](\eta)\, d\eta \\
&\quad + \int_0^b |T(t-s)| \int_0^s \left|f\left(\tau, y_n(\tau) + \hat{\phi}_\tau\right) - f\left(\tau, y(\tau) + \hat{\phi}_\tau\right)\right| d\tau\, ds \to 0.
\end{aligned}
$$

Thus, F is continuous. This completes the proof that F is completely continuous.

Finally the set $\zeta(F) = \{y \in C_b^0 : y = \lambda F y, \lambda \in (0,1)\}$ is bounded, since for every solution y in $\zeta(F)$ the function $x = y + \hat{\phi}$ is a mild solution of (3), for which we have proved that $\|x\|_1 \le K$, and hence,

$$\|y\|_1 \le K + \left\|\hat{\phi}\right\|.$$

Consequently, by Schaefer's theorem, the operator F has a fixed point in C_b^0. This means that any fixed point of F satisfying $(Fx)(t) = x(t)$ is a solution of problem (1) on J. Hence, system (1) is controllable on J.

4. EXAMPLE

Consider the following partial integrodifferential equation of the form

$$\begin{aligned} \frac{\partial}{\partial t}\left[z(y,t) - p(s, z(y,t-r))\right] &= \frac{\partial^2}{\partial y^2} z(y,t) + u(t) \\ + \int_0^t q(s, z(y, s-r))\,ds, &\qquad 0 \le y \le \pi, \qquad t \in J, \\ z(0,t) = z(\pi,t) = 0, &\qquad t \ge 0, \\ z(t,y) = \phi(y,t), &\qquad -r \le t \le 0, \end{aligned} \tag{4}$$

where ϕ is continuous and $u \in L^2(J,U)$ with $U \subset J$ and $X = L^2[0,\pi]$. Let $f(t,w_t)(y) = q(t, w(t-y))$, $0 \le y \le \pi$ and $g(t,w_t)(y) = p(t, w(t-y))$. Define $A : X \to X$ by $Aw = w''$ with domain $D(A) = \{w \in X,\ w, w' \text{ are absolutely continuous}, w'' \in X,\ w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, w \in D(A),$$

where $w_n(s) = \sqrt{2/\pi}\sin ns$, $n = 1,2,3,\dots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \ge 0$ in X and is given by

$$T(t)w = \sum_{n=1}^{\infty} \exp\left(-n^2 t\right)(w, w_n) w_n, w \in X,$$

where $T(t)$ satisfies hypothesis (i), since the analytic semigroup $T(t)$ is compact and $|T(t)| \le K$ for each $t \ge 0$ and $K > 0$.

Assume that there exists an invertible operator W^{-1} which takes values in $L^2[J,U]/\ker W$ such that

$$Wu = \int_0^b T(b-s) u(s)\,ds.$$

Further, the function $p : J \times [0,\pi] \to [0,\pi]$ is completely continuous and there exist constants $k_1 < 1$ and $k_2 > 0$ such that

$$\|p(t, w(t-y))\| \le k_1(\|w\|) + k_2,$$

and also there exists an integrable function $l : J \to [0,\infty)$ such that

$$\|q(t, w(t-y))\| \le l(t)\Omega_1(\|w\|),$$

where $\Omega_1 : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing. Also we have

$$\int_0^b \hat{n}(s)\,ds \le \int_c^{\infty} \frac{ds}{s + \Omega_1(s)},$$

where $c = (1/(1-k_1))[K(\|\phi\| + k_1\|\phi\| + k_2) + k_2 + K_1 k_2 b + KNb]$ and $\hat{n}(t) = \max\{(1/(1-k_1)) K_1 k_1, (K/(K_1 k_1))l(t)\}$. Here N depends on ϕ, p, and q.

Further, all the conditions of the above theorem are satisfied. Hence, system (4) is controllable on J.

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